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On the Problem of Penetration in Particle Methods

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A method is described which prevents penetration when particle methods are used to simulate streams of fluid impinging on each other. The method does not produce dissipation but it does produce extra dispersion.

1. INTRODUCTION

A technique for the elimination of penetration and mixing in particle methods like SPH [2, 3, 5] will be proposed and studied. The results may also have application to free-Lagrangian methods which are similar to SPH in that they do not use a grid, but determine the dynamics from information at a set of moving points.

The penetration we refer to occurs because SPH does not require that the velocity field be single valued. As a consequence, two or more particles, with different velocities, may occupy the same position. For low Mach number flows with smooth velocity fields this is seldom a problem, but in high Mach number flows the problem is severe. It can, however, be overcome by the use of an appropriate viscosity [1, 4] so that, in high Mach number collisions of gas clouds, the penetration is limited to about two resolution lengths.

The simulation of high Mach number flows is therefore in a fairly satisfactory state, but even there it would be a clear advantage, especially in problems involving re-expansion from a compressed interfacial state, to produce sharper interfaces. In subsonic flows the situation is far less satisfactory. Although, in general, opposing streams do not mix substantially, even in the absence of viscosity, mixing and disorder do occur. If the artificial viscosity is retained the disorder can be reduced, but at the price of decreasing the Reynold's number beyond acceptable limits.

It is natural to try and remedy this difficulty by modifying the interactions between particles. However, a series of experiments I performed using assignment functions (kernels) with different shapes designed to prevent particles mixing proved futile. They were often successful in one dimension but in two or three dimensions the particles can evade barriers by moving around each other. I discussed this problem with Jerry Brackbill who suggested that the procedure in FLIP

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[6] might be effective. In that method the rates of change of particle velocity, thermal energy, and position are determined from grid-based calculations. The result is that the velocity of a particle calculated from the momentum equation is not the velocity used for changing its position. Since the position-changing velocity is interpolated from the grid, the movement of the particles must vary smoothly across a cell and, in particular, no particles can penetrate because the velocity is single-valued. This idea, that the velocity from the momentum equation and the velocity used for changing position need not be identical, is the basis for the algorithm discussed in this paper.

2. The Equations of Motion

The SPH equations of motion for an inviscid fluid can be written [5]

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2}\right) \nabla_a W_{ab} \tag{2.1}$$

$$\frac{du_a}{dt} = \frac{1}{2}\sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2}\right) \mathbf{v}_{ab} \cdot \nabla_a W_{ab} \tag{2.2}$$

$$\frac{d\mathbf{r}_a}{dt} = \mathbf{v}_a \tag{2.3}$$

and the density ρ_a is determined either from

$$\frac{d\rho_a}{dt} = \sum_b m_b \mathbf{v}_{ab} \cdot \nabla_a W_{ab} \tag{2.4}$$

or from

$$\rho_a = \sum_b m_b W_{ab}. \tag{2.5}$$

In these equations m_b, $\mathbf{v}_b$, and u_b are the mass, velocity, and thermal energy per unit mass of particle b. The pressure P_b at particle b is determined through an equation of state depending on u_b and ρ_b. $W_{ab} \equiv W(\mathbf{r}_a - \mathbf{r}_b, h)$ is an interpolating kernel which is zero or negligible for $|\mathbf{r}_a - \mathbf{r}_b|$ greater than a few h, where h is a length that controls the width of W and the resolution of the numerical scheme. ∇_a denotes a gradient with respect to the coordinates of particle a, and $\mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b$. There are other possible forms for (2.1), but that shown has the advantage of simplicity and exact linear and angular momentum conservation if W is a spherically symmetric function. The energy equation (2.2) can also be written in different ways [5]. The density is usually calculated using (2.5) but there are some advantages in using (2.4). In this paper we use (2.1), (2.2), and (2.5) and modify (2.3).

The aim is to modify (2.3) so that particles that are close together will move with nearly identical velocities. In addition, we want linear and angular momentum to be conserved exactly, and we want the correction term to be galilean invariant. These aims are achieved by replacing (2.3) by

$$\frac{d\mathbf{r}_a}{dt} = \mathbf{v}_a + \sum_b \frac{m_b}{\bar{\rho}_{ab}} (\mathbf{v}_b - \mathbf{v}_a) W^*_{ab}, \tag{2.6}$$

where $\bar{\rho}_{ab} = \frac{1}{2}(\rho_a + \rho_b)$ and W^* is a kernel which need not be the same as W but we assume it is symmetric in a, b, and, like W, it has dimensions of 1/volume.

From (2.6) we can deduce immediately that

$$\sum_a m_a \frac{d\mathbf{r}_a}{dt} = \sum_a m_a \mathbf{v}_a, \tag{2.7}$$

which shows that the centre of mass moves correctly. Furthermore,

$$\sum_a m_a \mathbf{v}_a \times \frac{d\mathbf{r}_a}{dt} = 0 \tag{2.8}$$

and, since the momentum equation implies (assuming W_{ab} is symmetric in a, b) that

$$\sum m_a \mathbf{r}_a \times \frac{d\mathbf{v}_a}{dt} = 0 \tag{2.9}$$

it follows that angular momentum is conserved. A simple result worth noting is that a bunch of particles each with the same constant velocity will be unaffected by changing (2.3) to (2.6).

On first meeting the replacement of (2.3) by (2.6) the usual first reaction is that it is a very strange thing to do (which may be true) and the second reaction is that it must produce dissipation (which is certainly wrong). To make it touch more familiar ground we now examine its effect on wave propagation.

3. Wave Propagation in One Dimension

We consider an infinite line of particles initially in static equilibrium. We assume the particles have equal mass m and are equi-separated. We use the distance between nearest neighbours as the unit of length, and to simplify the analysis we assume the gas is isothermal and write $P = \rho c^2$, where c is a constant. The position of particle j is then j and its initial density is $\rho = m$.

To study small oscillations we consider time variations $\exp(-i\omega t)$ and Fourier analyse the spatial variations:

$$x_a = a + qe^{i\kappa a}, \qquad \rho_a = \rho + re^{i\kappa a}, \qquad \text{and} \qquad v_a = ve^{i\kappa a}. \tag{3.1}$$

Linearizing (2.1) and (2.5) we get

$$-i\omega v_a = -mc^2 \sum_b \left[\frac{2}{\rho} (\delta x_a - \delta x_b) \frac{\partial^2 W_{ab}}{\partial x_a^2} - \frac{(\delta \rho_a + \delta \rho_b)}{\rho^2} \frac{\partial W_{ab}}{\partial x_a} \right] \tag{3.2}$$

and

$$\delta \rho_a = m \sum_b (\delta x_a - \delta x_b) \frac{\partial W_{ab}}{\partial x_a}. \tag{3.3}$$

From (3.3) and (3.1)

$$r = mq \sum_b (1 - e^{i\kappa(b-a)}) \frac{\partial W_{ab}}{\partial x_a}. \tag{3.4}$$

To keep the analysis simple, without losing essential features, we replace summations by integrations according to the rule

$$\sum g(a-b) \to \int_{-\infty}^{\infty} g(u)\, du, \tag{3.5}$$

and we use the gaussian kernel

$$W_{ab} = \frac{1}{h\sqrt{\pi}} \exp[-(a-b)^2/h^2]. \tag{3.6}$$

Relation (3.4) then becomes

$$r = -i\kappa mq e^{-h^2\kappa^2/4}. \tag{3.7}$$

Applying these results and approximations to (3.2) we find

$$-i\omega v = -2qc^2\kappa^2 e^{-h^2\kappa^2/4} + \frac{c^2 r}{\rho} i\kappa e^{-h^2\kappa^2/4}. \tag{3.8}$$

Substituting for r in (3.8) using (3.7) gives

$$-i\omega v = -qc^2\kappa^2[2e^{-h^2\kappa^2/4} - e^{-h^2\kappa^2/2}]. \tag{3.9}$$

To complete the analysis we need the relation between position and velocity. If we use the standard relation (2.3) then

$$-i\omega q = v, \tag{3.10}$$

and substitution in (3.9) gives the dispersion relation

$$\begin{aligned} \omega^2 &= c^2\kappa^2[2e^{-h^2\kappa^2/4} - e^{-h^2\kappa^2/2}] \\ &= c^2\kappa^2 + O(\kappa^6) \qquad \text{for} \quad h\kappa \ll 1. \end{aligned} \tag{3.11}$$

If, instead of (2.3), we use (2.6) with $W^* = W$, (3.10) is replaced by

$$-i\omega q = v[1 + (e^{-h^2\kappa^2/4} - 1)], \tag{3.12}$$

where the term, $(\exp(-h^2\kappa^2/4) - 1)$, comes from the summation in (2.6). The dispersion relation is now

$$\begin{aligned}\omega^2 &= c^2\kappa^2[2e^{-h^2\kappa^2/2} - e^{-3h^2\kappa^2/4}] \\ &= c^2\kappa^2 + O(\kappa^4) \qquad \text{for} \quad h\kappa \ll 1.\end{aligned} \tag{3.13}$$

The use of (2.6) instead of (2.3) increases the dispersion without introducing any dissipation.

The absence of dissipation can be deduced in a more fundamental way by noting that the term added to $\mathbf{v}_a$ in (2.6) does not change the time parity, i.e., replacing t by $-t$ and $\mathbf{v}$ by $-\mathbf{v}$ leaves the equations of motion invariant.

Because the dispersion relation is altered so is the group velocity. From (3.13) we find that the group velocity for long wavelengths is reduced, and information is therefore transmitted more slowly than when (2.3) is used.

This relationship between information transfer and the position-changing rule can be seen easily in an extreme case. Suppose two people exchange information across an ice rink by writing messages on pucks and projecting them towards each other at constant velocity (this is a thought experiment and we allowed to assume zero viscosity!). If (2.6) was a law of nature, and two pucks (with diameter $\ll h$) passed within h of each other, they would move more slowly and the messages would therefore be transmitted more slowly than if (2.3) applied. No dissipation would occur because there are no forces, and the momentum equation informs us that the momentum of any puck is constant.

An approximation to (2.6) useful in qualitative discussions of one dimensional motion is obtained by expanding $v_b - v_a$ in a Taylor series and changing the summation to integration. We find the approximate equation

$$\frac{dx_a}{dt} \simeq v_a + \frac{1}{2}\frac{\partial^2 v_a}{\partial x^2}\int u^2 W(u)\, du. \tag{3.14}$$

Since, if $W(u, h)$ is an even function of u, and $W \geqslant 0$,

$$\int u^2 W(u)\, du = \beta h^2, \tag{3.15}$$

where β is a positive constant (3.14) can be written

$$\frac{dx_a}{dt} \simeq v_a + \frac{1}{2}\beta h^2 \frac{\partial^2 v_a}{\partial x^2}. \tag{3.16}$$

The effect of (2.6) on a Fourier mode with $\kappa h < 1$ is easily deduced from (3.16).

It has already been pointed out that W^* in (2.6) need not be the same as W. We can use this flexibility to improve the dispersion properties for long wavelength modes.. For example if we choose W^* to be super-gaussian

$$W(u, h) = \frac{1}{h\pi^{1/2}} e^{-u^2/h^2} \left(\frac{3}{2} - \frac{u^2}{h^2}\right), \tag{3.17}$$

for which

$$\int_{-\infty}^{\infty} W(u, h) e^{i\kappa u}\, du = e^{-h^2\kappa^2/4} \left(1 + \frac{h^2\kappa^2}{4}\right), \tag{3.18}$$

then (3.12) is replaced by

$$-i\omega q = ve^{-h^2\kappa^2/4} \left(1 + \frac{h^2\kappa^2}{4}\right). \tag{3.19}$$

and the dispersion relation becomes

$$\begin{aligned} \omega^2 &= c^2\kappa^2 [2e^{-h^2\kappa^2/2} - e^{-3h^2\kappa^2/4}] \left(1 + \frac{h^2\kappa^2}{4}\right) \\ &= c^2\kappa^2 + O(\kappa^6) \qquad \text{if} \quad h\kappa \ll 1. \end{aligned} \tag{3.20}$$

which is similar, for long wavelengths, to (3.11). For this kernel relation (3.16) is replaced by

$$\frac{dx_a}{dt} \simeq v_a - \gamma h^4 \frac{\partial^4 v}{\partial x^4}, \tag{3.21}$$

where γ is a positive constant. The super-gaussian kernel therefore ameliorates the effect of (2.6) on smooth flows. If the velocity field changes rapidly with position, as will occur near the interface between colliding fluids, (2.6) with the super-gaussian kernel will still act to prevent penetration.

4. Time Stepping

There are many possible forms of time stepping. For the experiments described in the text section a simple predictor-corrector scheme, which conserves linear and angular momentum exactly, is used. To describe this scheme it is convenient to write Eqs. (2.1) and (2.2) as

$$\frac{d\mathbf{v}_a}{dt} = \mathbf{F}_a \qquad \text{and} \qquad \frac{du_a}{dt} = Q_a, \tag{4.1}$$

and Eq. (2.6) as

$$\frac{d\mathbf{r}_a}{dt} = \mathbf{v}_a + \Delta\mathbf{v}_a =: \mathbf{V}_a, \tag{4.2}$$

where $\Delta\mathbf{v}_a$ denotes the summation term in (2.6). Superscripts 0, 1 denote current and new values, respectively. We predict according to

$$\bar{\mathbf{v}}_a^1 = \mathbf{v}_a^0 + \delta t\, \mathbf{F}_a^0; \qquad \bar{u}_a^1 = u_a^0 + \delta t\, Q_a^0,$$

and (4.3)

$$\bar{\mathbf{r}}_a^1 = \mathbf{r}_a^0 + \delta t\, \mathbf{V}_a.$$

Values of ρ, F, and Q at the midpoint are then estimated using

$$\mathbf{r}_a^{1/2} = \tfrac{1}{2}(\bar{\mathbf{r}}_a^1 + \mathbf{r}_a^0)$$

and similar expressions for $\mathbf{v}_a^{1/2}$ and $u_a^{1/2}$ (which with $\rho_a^{1/2}$, and the equation of state, gives $P_a^{1/2}$). We then correct according to

$$\mathbf{v}_a^1 = \mathbf{v}_a^0 + \delta t\, \mathbf{F}_a^{1/2}, \qquad u_a^1 = u_a^0 + \delta t\, Q_a^{1/2}, \tag{4.4}$$

and

$$\mathbf{r}_a^1 = \mathbf{r}_a^0 + \tfrac{1}{2}\delta t(\mathbf{V}_a^1 + \mathbf{V}_a^0). \tag{4.5}$$

Because of the form of $\mathbf{F}_a$ it is easy to see that

$$\sum_a m_a \mathbf{v}_a^1 = \sum_a m_a \mathbf{v}_a^0,$$

and linear momentum is conserved. Furthermore, since

$$\sum_a m_a \mathbf{r}_a^{1/2} \times \mathbf{F}_a^{1/2} = 0, \tag{4.6}$$

$$\mathbf{r}_a^1 = \mathbf{r}_a^{1/2} + \tfrac{1}{2}\delta t\, \mathbf{V}_a^1 \qquad \text{and} \qquad \mathbf{r}_a^{1/2} = \mathbf{r}_a^0 + \tfrac{1}{2}\delta t\, \mathbf{V}_a^0, \tag{4.7}$$

while

$$\sum_a m_a \mathbf{v}_a^1 \times \mathbf{V}_a^1 = 0 = \sum_a m_a \mathbf{v}_a^0 \times \mathbf{V}_a^0, \tag{4.8}$$

we infer from the first of (4.4) and (4.6) that

$$\sum_a m_a \mathbf{r}_a^{1/2} \times \mathbf{v}_a^1 = \sum m_a \mathbf{r}_a^{1/2} \times \mathbf{v}_a^0. \tag{4.9}$$

The first of (4.7) and (4.8) reduce the LHS of (4.9) to

$$\sum m_a \mathbf{r}_a^1 \times \mathbf{v}_a^1,$$

and the second of (4.7) and (4.8) reduce the RHS of (4.9) to

$$\sum m_a \mathbf{r}_a^0 \times \mathbf{v}_a^0,$$

and the conservation of angular momentum is established. In practice a variant of the above procedure halves the computation time. Instead of predicting with $\mathbf{F}_a^0$ and Q_a^0 we use the midpoint values from the previous time step. This introduces errors of $O(\delta t)^3$ in the final values of $\mathbf{v}$. The overall algorithm has second-order errors in time and space (i.e., the errors are $O(\delta t)^2$ and $O(h)^2$).

When we wish to describe shock phenomena we use an artificial viscosity which involves adding to

$$P_a/\rho_a^2 + P_b/\rho_b^2,$$

in the momentum equation and energy equation the term

$$(\alpha \zeta_{ab} \bar{c}_{ab} + \beta \zeta_{ab}^2)/\bar{\rho}_{ab}. \tag{4.10}$$

In (4.10), $\bar{c}_{ab} = \frac{1}{2}(c_a + c_b)$, where c_a is the speed of sound at particle a, $\bar{\rho}_{ab} = \frac{1}{2}(\rho_a + \rho_b)$, and if $\mathbf{v}_{ab} \cdot \mathbf{r}_{ab} < 0$,

$$\zeta_{ab} = -\frac{(\mathbf{v}_{ab} \cdot \mathbf{r}_{ab})h}{r_{ab}^2 + 0.01h^2}; \tag{4.11}$$

otherwise $\zeta_{ab} = 0$. In (4.11), $\mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b$ and $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$.

The usual form of an SPH program requires a subroutine to calculate the ρ's and a subroutine to calculate the F's and Q's. These subroutines require about 80% of the total CPU in a fluid dynamics program without self gravity. Since we require $\Delta \mathbf{v}_a^1$ in (4.5) an extra subroutine similar to that for ρ must be used. This increases the computation time by about 30%.

The rule for the time step δt is to choose

$$\delta t = 0.3 \operatorname{Min}(\delta t_1, \delta t_2) \tag{4.12}$$

where

$$(\delta t_1) = \operatorname*{Min}_a (h/\sqrt{|\mathbf{F}_a|})$$

and

$$\delta t_2 = \operatorname*{Min}_a (h/(c_a + 1.2\alpha c_a + 1.2\beta \operatorname*{Max}_b \zeta_{ab})).$$

The form of δt_2 is obtained by combining the Courant condition with stability conditions derived by observing that the artificial viscosity produces diffusion.

5. Numerical Tests

The tests described here use a 2-dimensional form of SPH. The first test (a) compares the penetration resulting from two identical streams meeting head on when (2.3) and (2.6) are used. Test (b) shows that when (2.6) is used shock phenomena are still treated correctly. Test (c) was designed to determine if the use of (2.6) gave false results for an astrophysical problem where symmetry and angular momentum should be preserved. The tests use the kernel based on $\mathbf{M}_4$ splines [5] which has continuous second derivatives.

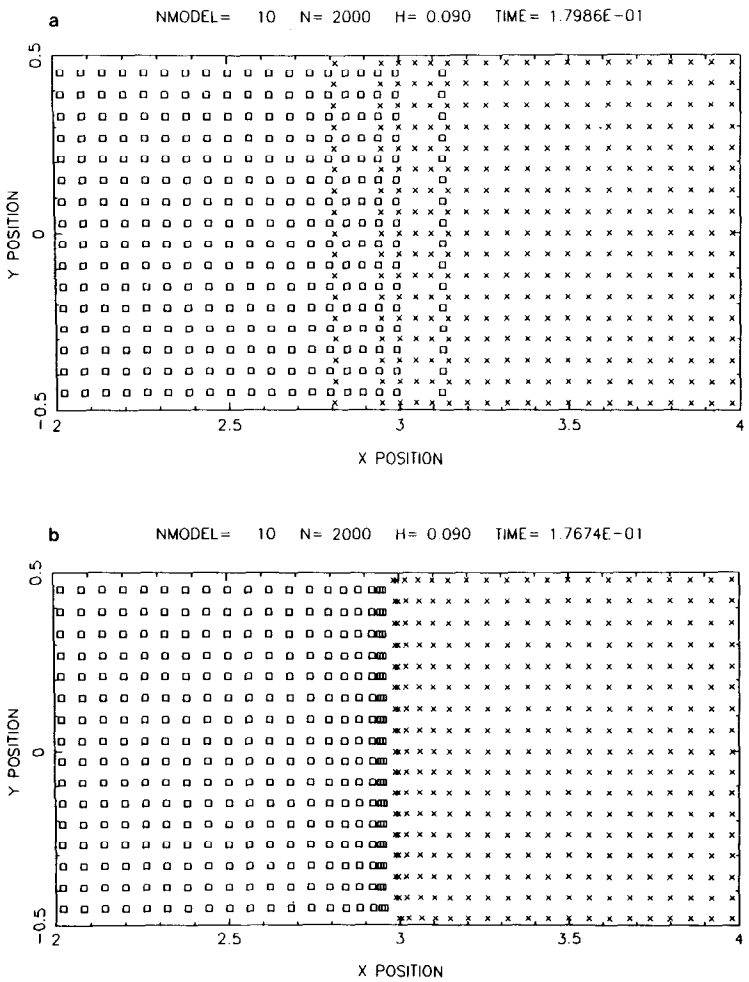

FIG. 1. The collision of two streams each with speed = Mach 1, and each with initial density = 1. (a) shows the results using standard SPH with no artificial viscosity. (b) shows the results when (2.6) is used.

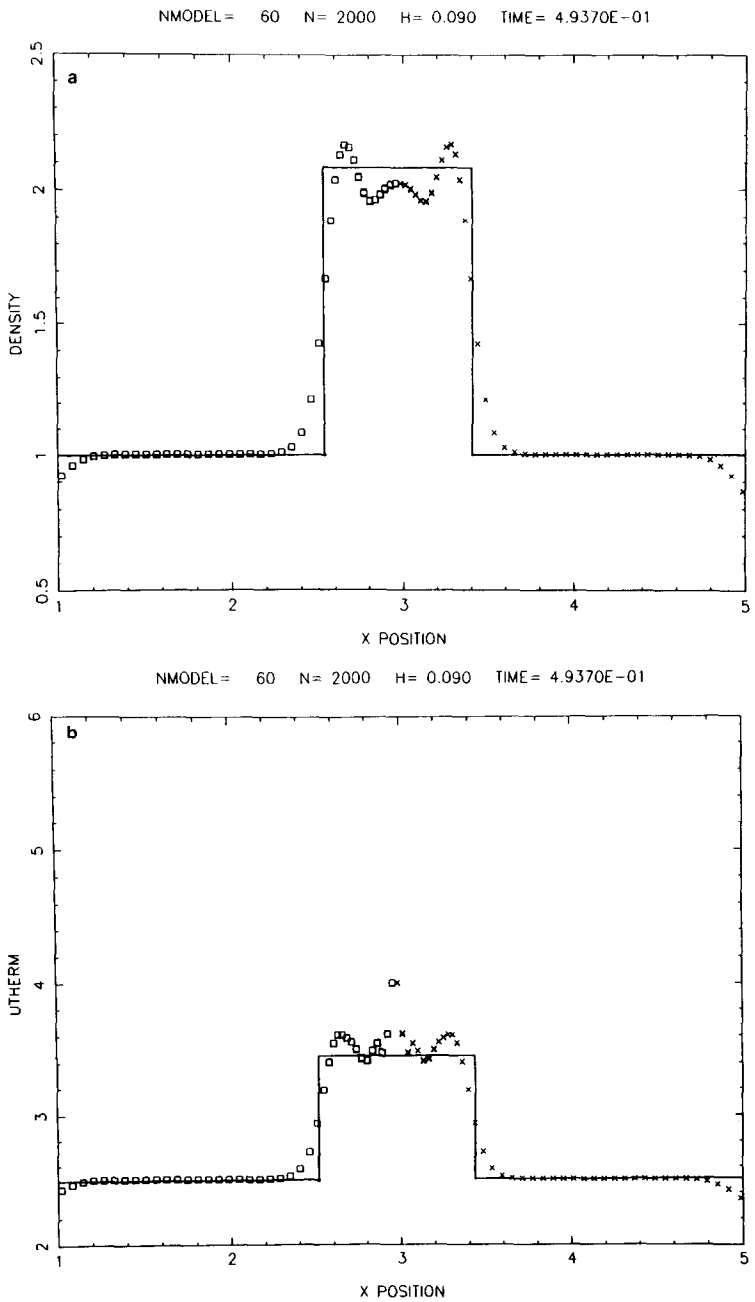

FIG. 2. The same configuration as in Fig. 1 with (2.6) and an artificial viscosity ($\alpha = 1$, $\beta = 2$). The exact results are shown by continuous lines. The edge effects arise because the streams are of finite size: (a) density, (b) thermal energy, and (c) velocity. "Wall heating" is the cause of the peak in the thermal energy and the resulting dip in the density.

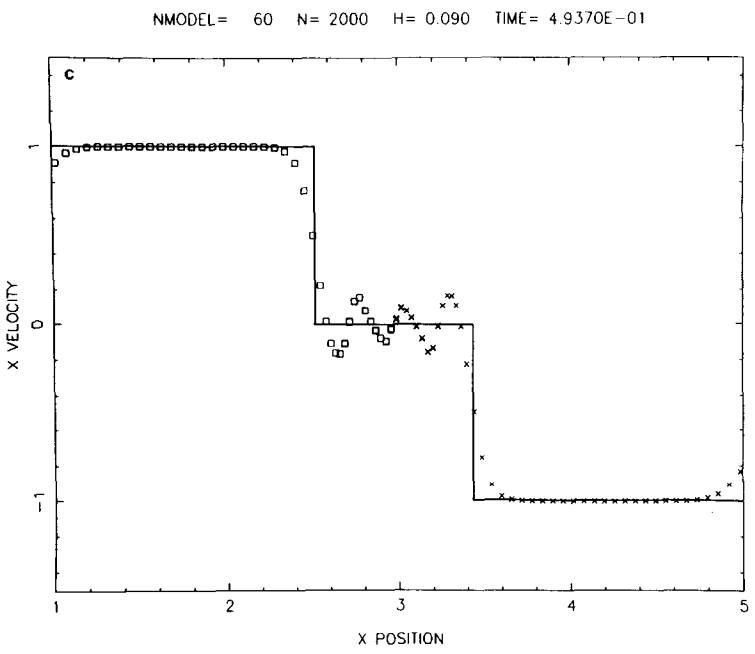

FIGURE 2—*Continued*

If $0 < r < h$ then

$$h^2 W = \frac{15}{7\pi}\left(\frac{2}{3} - \left(\frac{r}{h}\right)^2 + \frac{1}{2}\left(\frac{r}{h}\right)^3\right)$$

else, if $h < r < 2h$,

$$h^2 W = \frac{15}{42\pi}\left(2 - \frac{r}{h}\right)^3$$

else

$$h^2 W = 0.$$

(a) *Penetration in collisions*

In this test there are two streams. For $x < 3$ the stream initially has velocity $= 1$ (Mach 1) and $\rho = 1$ while for $x > 3$ the velocity $= -1$ and $\rho = 1$. The particles have equal mass, they are separated initially by 0.06 and $h = 0.09$. The initial thermal energy is $1/(\gamma - 1)$ and $\gamma = 1.4$. The particles for $x < 0$ are displaced in the y direction to make the test more demanding. The artificial viscosity is zero.

In the first frame of Fig. 1 we show the results when (2.3) is used. There is substantial penetration. In the second frame (2.6) has been used. There is now no penetration. Experiments with initial velocities up to Mach 10 give the same result.

(b) *Shocks*

The simplest shock configuration is the same as for (a). We use the viscosity given by (4.10) with $\alpha = 1$, $\beta = 2$. The results are shown in Fig. 2, and they are similar to those found when (2.3) is used. No attempt has been made to fine tune the viscosity since the appropriate form of viscosity to use with (2.6) is an open question. The results are in satisfactory agreement with the exact results and are similar to those obtained by the Lax–Wendroff method. However, to the reader used to the high accuracy which can be achieved for shocks using TVD or other monotonic schemes, these results will appear crude. The ripples on the profiles may be due to the initial configuration since experiments on a steady shock, using SPH, show that setting up the theoretically exact discontinuous initial conditions gives large ripples, but simulating a piston moving into the gas (which theoretically gives the same profile) gives a smooth profile with no ripples.

Similar accuracy was found for flows initially at Mach 10.

(c) *Particles in Orbit*

A typical problem that can be tackled with SPH is the evolution of low viscosity disks in orbit about a central mass. In this problem it is important that, in the absence of pressure forces, the solution technique guarantees that the particles will move on accurate orbits. This can be tested by setting up rings of particles with initial conditions which should keep each particle in a circular orbit. Errors arising from (2.6) will show up in the distortion of the rings. For the test described 20 rings each of 100 particles are used. Pressure and artificial viscosity were turned off. The momentum equation then takes the form

$$\frac{d\mathbf{v}_a}{dt} = -\frac{\mathbf{r}_a}{r_a^3},$$

where we assume the central mass = 1. Density is calculated using (2.5), and (2.6) is used to change the particle positions. The inner ring is at radius = 1, and the rings are separated by 0.03; $h = 0.09$ and the particles have equal mass. The density for each particle at an early stage of the motion is shown in the first frame of Fig. 3 and, after one rotation of the inner ring, in the second frame, two points emerge. First, since each density symbol is the superposition of the density symbol of 100 particles in one ring, the rings remain highly axisymmetric. Loss of symmetry would show up as fuzziness and broadening. The second point is that the density varies only very slightly. The maximum effect is on the outer edge. This is a good result because it might have been expected that, with the density falling off rapidly near the edge, something untoward might have happened to the summation term in (2.6). In Fig. 4 the particle positions are shown. The positions after one revolution of the inner ring have a curious appearance because of the patterns which form when a particle catches up to the particle in front of it in the adjacent outer ring. When the printer plots these points, it plots the points on one of the original radial lines, and the expected spiral of this line due to differential rotation is then clear.

These results show that (2.6) does not degrade the orbits of particles significantly.

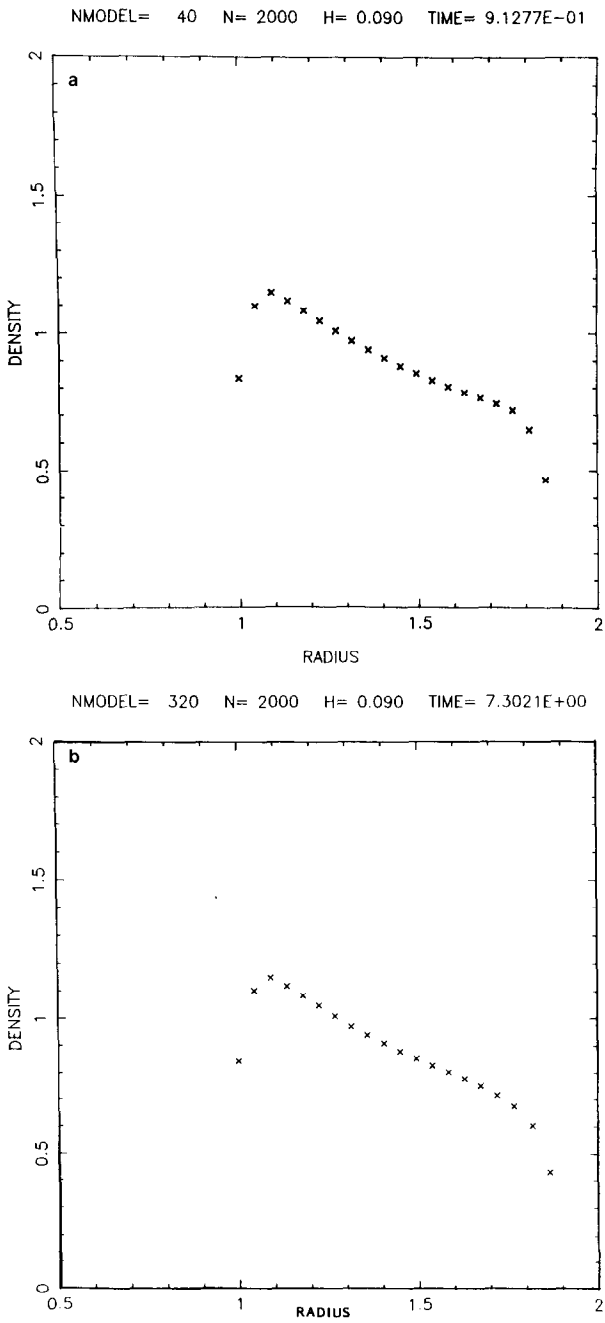

FIG. 3. The variation of density with radius for the ring configuration described in the text: (a) early stage; (b) just after one full rotation of the inner ring. Each symbol is the superposition of the density for the 100 particles in each ring and the absence of blurring shows accurate axial symmetry.

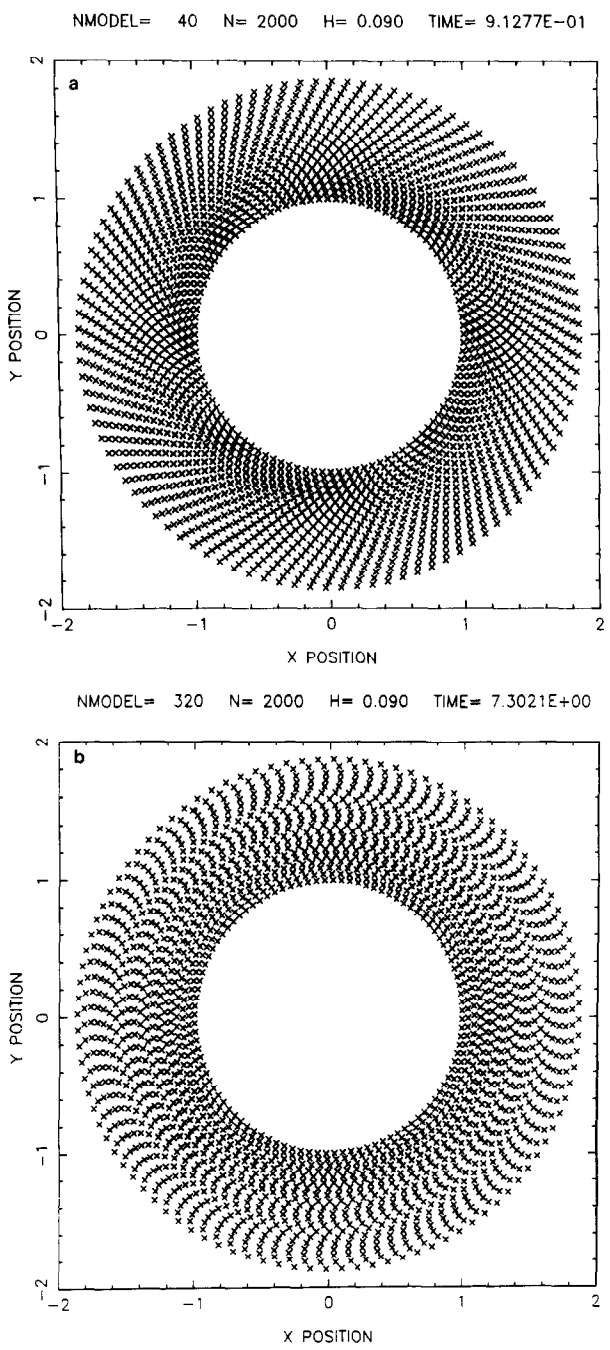

FIG. 4. The particle configuration for the rings producing the density illustrated in Fig. 3: (a) early stage; (b) after one full rotation of the inner ring.

6. Discussion and Conclusions

The simple technique described here prevents penetration without dissipation and linear and angular momentum are conserved. It has negligible effect on motion in a gravitational field and it does not degrade the simulation of shocks. The technique is clearly promising although there are a number of remaining questions concerning its application.

The first of these is the correct way to combine (2.6) with an artificial viscosity. Since (2.6) makes the particle method more nearly like the representation of a fluid we are in the same situation as the user of finite difference methods. It may therefore be possible to use some of the standard artificial viscosities which, for standard SPH, do not give good results [1].

The second question concerns the ideal way to introduce the position-changing velocity into the thermal energy equation. The results described in this paper use the momentum velocity in the thermal energy equation which gives excellent energy conservation, but is inconsistent with the way position is changed since that determines $d\rho/dt$ and, in turn, du/dt. Because the results we have found are already quite satisfactory the precise answer to this last question may not be important.

Acknowledgments

This work was begun at the Los Alamos National Laboratory where Jerry Brackbill's results from FLIP were a constant and stimulating challenge. It was completed at the Institute of Astronomy, Cambridge, UK, where the disbelieving stares I got when I described (2.6) forced me to try to understand what was really going on.

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